INVERTIBILITY THRESHOLD FOR H^∞ TRACE ALGEBRAS, AND EFFECTIVE MATRIX INVERSIONS

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Dedicated to the memory of M. S. Birman, from whom both of us were learned a lot (and not only mathematics)

Abstract. For a given δ, $0 < \delta < 1$, a Blaschke sequence $\sigma = \{\lambda_j\}$ is constructed such that every function f, $f \in H^\infty$, having $\delta < \delta_f = \inf_{\lambda\in\sigma} |f(\lambda)| \le \|f\|_\infty \le 1$ is invertible in the trace algebra $H^\infty|\sigma$ (with a norm estimate of the inverse depending on δ_f only), but there exists f with $\delta = \delta_f \le \|f\|_\infty \le 1$, which does not. As an application, a counterexample to a stronger form of the Bourgain–Tzafriri restricted invertibility conjecture for bounded operators is exhibited, where an "orthogonal (or unconditional) basis" is replaced by a "summation block orthogonal basis".

1. Introduction

The paper deals with a numerical control of inverses (condition numbers) for functions $T = f(A)$ of large matrices in terms of the lower spectral parameter

$$\delta = \delta(T) = \min |\lambda_j(T)|$$

Precisely, our problem is the following. Given a sequence $\sigma = \{\lambda_j\}$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ of the complex plain, we consider all normalized matrices A, $\|A\| \le 1$ (or Hilbert space operators) such that $\sigma(A) \subset \sigma$ (counting multiplicities) and look for a numerical function $c(\delta) = c(\delta, \sigma)$ bounding the inverses

$$\| T^{-1}\| \le c(\delta)$$

for all $T = f(A)$ having $\delta \le |\lambda_j(T)| \le \|T\| \le 1$, where $\lambda_j(T)$ mean eigenvalues of $T = f(A)$. The best possible upper bound $c(\delta)$ is called

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$c_1(\delta) = c_1(\delta, \sigma)$,

$$c_1(\delta, \sigma) = \sup\left\{\|T^{-1}\| \colon T = f(A), \delta \le |\lambda_j(T)| \le \|T\| \le 1,\ \sigma(A) \subset \sigma,\ \|A\| \le 1\right\}.$$

Here f can be a polynomial (if A is a finite matrix) or an H^∞ function (if A is a Hilbert space contraction). Recall that

$$H^\infty = \left\{ f \colon f \text{ holomorphic on } \mathbb{D} \text{ and } \|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)| < \infty \right\}.$$

Since $\delta \mapsto c_1(\delta, \sigma)$, $0 < \delta < 1$, is a decreasing function, we can define a *critical constant* (or, an *invertibility threshold*) $\delta_1 = \delta_1(\sigma)$, $0 \le \delta_1 \le 1$, by the following properties

$$\begin{aligned} 0 < \delta < \delta_1 &\implies c_1(\delta) = \infty\,, \\ \delta_1 < \delta \le 1 &\implies c_1(\delta) < \infty\,. \end{aligned}$$

The number δ_1 can be considered as a threshold of bounded invertibility or as a threshold for an operator algebra to be *inverse closed*: operators T from our collection with a "*scattered*" spectral data (i. e., $\inf_j |\lambda_j(T)| < \delta_1$, $\|T\| = 1$) are, in general, not invertible, whereas those with "*flat*" spectral data $\delta_1 < \delta \le |\lambda_j(T)| \le \|T\| \le 1$ are invertible.

The principal result of this paper is a construction of a Blaschke sequence σ with a given in advance value of the critical constant $\delta_1(\sigma) = \delta_1$, $0 \le \delta_1 \le 1$ (Section 2 below).

The case, where $\delta_1 = 0$, was considered in [GMN]; moreover, the paper quoted contains necessary and sufficient conditions for $\delta_1(\sigma) = 0$, which reduces to the so-called Weak (Carleson) Embedding Property (WEP). See the statement of the result at the end of this Introduction.

It is worth mentioning that, strictly speaking, the properties of a function algebra A on a set σ to be inverse closed (i. e., the property $f \in A$, $\inf_{z \in \sigma} |f(z)| > 0 \implies 1/f \in A$) does not imply that $\delta_1(\sigma, A) = 0$ (this fact was already mentioned in [GMN]; the constants $c_1(\sigma, A)$ and $\delta_1(\sigma, A)$ are defined for an algebra A in a similar way). Indeed, for an arbitrary Blaschke sequence $\sigma = \{\lambda_j\}$, the trace algebra $A = C_a(\mathbb{D})|\sigma$ of the disk algebra $C_a(\mathbb{D}) = H^\infty \cap C(\overline{\mathbb{D}})$, is always inverse closed, whereas $c_1(\delta, C_a(\mathbb{D})|\sigma) = c_1(\delta, H^\infty|\sigma)$ for every δ, $0 < \delta < 1$, and hence $\delta_1(\sigma, C_a(\mathbb{D})|\sigma) = \delta_1(\sigma, H^\infty|\sigma)$, but the algebra $H^\infty|\sigma$ can be not inverse closed (i. e. possibly $\delta_1(\sigma, C_a(\mathbb{D})|\sigma) > 0$). These properties are shown in [GMN].

The constant $c_1(\delta, \sigma)$ has a meaning of "the best estimate for the worst case" when bounding inverse matrices in terms of the lower spectral parameter δ. Moreover, we can describe it in two more ways, at

least in the case of the "simple spectrum" (the points λ_j of the sequence σ are pairwise different). First, it is the optimal upper bound for inverses in the *trace algebra*

$$H^\infty|\sigma = \{a\colon \sigma \to \mathbb{C}\colon \exists f \in H^\infty \text{ such that } a = f|\sigma\}$$

endowed with the trace norm $\|a\| = \inf\{\|f\|_\infty\colon a = f|\sigma\}$. Examples of such algebras with a given threshold δ_1 of the bounded invertibility (Section 2 below) are, probably, of interest for the H^∞ interpolation theory.

Secondly, in the definition of $c_1(\delta,\sigma)$, we can restrict ourselves to a just one ("the worst") contraction A and the algebra generated by H^∞ functions of it. This is the so-called *model contraction* M_B, which can be defined as follows. Given a Blaschke product $B = B_\sigma$

$$B_\sigma = \prod_{j\ge 1} b_{\lambda_j}\,,$$

where $b_\lambda = \frac{\lambda - z}{1-\bar\lambda z}\cdot\frac{|\lambda|}{\lambda}$, $\lambda \in \mathbb{D}$, and $\sigma = \{\lambda_j\}$, $\sum_j (1-|\lambda_j|) < \infty$ (the Blaschke condition), we set

$$M_B^* f = \frac{f - f(0)}{z}\,, \qquad f \in K_B\,,$$

where $K_B = H^2 \ominus BH^2$ (the orthogonal complement of BH^2 in H^2) and H^2 stands for the standard Hardy space of the disk,

$$H^2 = \Big\{ f = \sum_{k\ge 0} a_k z^k\colon \sum_{k\ge 0} |a_k|^2 = \|f\|_2^2 < \infty \Big\}.$$

It is well known (and easy to verify, see [Nik1], [Nik2]) that

$$\begin{aligned} M_B K_B &\subset K_B,\\ \|M_B\| &= 1,\\ \text{and}\quad \sigma(M_B) &= \operatorname{clos}\{\lambda_j\colon j = 1,2,\dots\}. \end{aligned}$$

Moreover, $\|M_B^{-1}\| = 1/B(0)$. It is also known that for every matrix A with $\|A\| \le 1$ and $\sigma(A) \subset \sigma$, one has $\|f(A)\| \le \|f(M_B)\|$ for every function f. This entails that the question on the invertibility and the norm control of inverses can be reduced to functions $f(M_B)$ of the model operator only, and inverse $f(M_B)^{-1}$, if it exists, is again a H^∞-function of M_B.

The above discussion easily implies the following.

(1) If the set $\{\lambda \in \sigma\colon \delta \le |\lambda| \le \delta'\}$ is infinite for some $0 < \delta \le \delta' < 1$, then $c_1(\delta,\sigma) = \infty$.

(2) If σ is a sequence tending to the unit circle (i. e., $\{\lambda \in \sigma : |\lambda| \le \delta\}$ is finite for every $\delta < 1$) and $\sum_{\lambda\in\sigma}(1-|\lambda|) = \infty$, then $c_1(\delta,\sigma) = \infty$ for every δ, $0 < \delta < 1$.

These properties show that, in fact, *the Blaschke condition is necessary in order our questions* (to find or to estimate $c_1(\delta,\sigma)$ and $\delta_1(\sigma)$) *to be nontrivial*. In what follows we always assume this property (if the converse does not stated explicitly). Now, we can give the following expression for $c_1(\delta,\sigma)$.

Lemma 1. *Let σ be a Blaschke subset of the unit disk $\mathbb{D}$. Then*

$$c_1(\delta,\sigma) = c_1(\delta, H^\infty|\sigma) = c_1(\delta, H^\infty/BH^\infty)$$

for every δ, $0 < \delta < 1$, where $B = B_\sigma$ and

$$\begin{aligned} c_1(\delta, H^\infty/BH^\infty) =: \sup &\left\{ \left\|\frac{1}{f}\right\|_{H^\infty/BH^\infty} : \|f\|_\infty \le 1, \delta \le \|f(\lambda)\| \text{ for } \lambda\in\sigma \right\} \\ = \sup &\Big\{ \inf\big[\|g\|_\infty : gf + hB = 1\big] : \\ &\qquad \delta \le \|f(\lambda)\| \le \|f\|_\infty \le 1 \text{ for } \lambda \in \sigma \Big\} \end{aligned}$$

and $\|h\|_{H^\infty/BH^\infty}$ means $\inf\{\|g\|_\infty : g(\lambda) = h(\lambda)$ *for* $\lambda\in\sigma\}$.

Proof. For every matrix A, $\|A\| \le 1$, and $f \in H^\infty$, the von Neumann inequality entails

$$\|f(A)\| \le \|f\|_\infty .$$

Since $B(A) = 0$, $B = B_\sigma$, for A having $\sigma(A) \subset \sigma$, we get

$$\|f(A)\| \le \inf_{g\in H^\infty} \|f + Bg\|_\infty = \|f\|_{H^\infty/BH^\infty} .$$

This implies $f(A)^{-1} = h(A)$ and $\|f(A)^{-1}\| \le \|h\|_{H^\infty}$ for every solution h of the equation $fh + Bk = 1$, and therefore $\|f(A)^{-1}\| \le \|\frac{1}{f}\|_{H^\infty/BH^\infty}$. Thus, $c_1(\delta,\sigma) \le c_1(\delta, H^\infty/BH^\infty)$.

On the other hand, there exists an "extreme operator" (matrix) for which the above calculus inequality becomes an identity. Indeed, if $A = M_B$, the "model operator" mentioned above, then $\|h(M_B)\| = \|h\|_{H^\infty/BH^\infty}$ for every $h \in H^\infty$ (Sarason's commutant lifting theorem, see for example, [Nik1] or [Nik2]). Hence, $c_1(\delta,\sigma) \ge c_1(\delta, H^\infty/BH^\infty)$. □

Finally, we quote the principal result from [GMN]

Theorem 2. ([GMN]) *Let $\sigma = \{\lambda_j\}$ be a Blaschke sequence in the disk $\mathbb{D}$. The following are equivalent.*

(1) $\delta_1(H^\infty|\sigma) = 0$.

(2) *The following Weak Embedding Property holds*: *for every* $\varepsilon > 0$ *there exists* C *such that*

$$\sum_{j\geq 1} \frac{(1-|\lambda_j|^2)(1-|z|^2)}{|1-\bar{\lambda}_j z|^2} \leq C$$

for every $z \in \mathbb{D} \setminus \bigcup_{\lambda\in\sigma} \left\{\zeta \colon |b_\lambda(\zeta)| < \varepsilon\right\}$.

(3) *For every* $\varepsilon > 0$ *there exists* η *such that* $|B(z)| \leq \eta$ *implies* $\inf_{\lambda\in\sigma} |b_\lambda(z)| \leq \varepsilon$; *here* B *is the corresponding Blaschke product* $B = \prod_{\lambda\in\sigma} b_\lambda$.

Moreover, if $\eta(\varepsilon) = \max\{\eta\}$ *over all* η *admitted in* (3), *then*

$$\frac{1}{\eta(\delta)} \leq c_1(\delta, H^\infty|\sigma) \leq \frac{a}{\eta(\delta/3)^2} \log \frac{1}{\eta(\delta/3)}$$

for every δ, $0 < \delta < 1$; $a > 0$ *is a numerical constant.*

The paper is organized as follows. Section 2 contains our principal result: for a given δ, $0 < \delta < 1$, there exists a Blaschke product $B = B_\sigma$ such that $\delta_1(\sigma, H^\infty|\sigma) = \delta$. We also exhibit an upper estimate for $c_1(\delta, H^\infty|\sigma)$ for $\delta_1 < \delta \leq 1$. Since the problem (and our result) on the invertibility threshold is conformally invariant, we will change the variable and work (in Section 2) in the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} \colon \operatorname{Im}(z) > 0\}$ instead of the unit disk $\mathbb{D}$.

In Section 3, we use the above result in order to give a counterexample to a stronger form of the so-called (Bourgain–Tzafriri) restricted invertibility conjecture. The conjecture claims (see [CCLV] and comments in Section 3): for every unconditional normalized basic sequence $\{x_j\}_{j\in J}$ in a Hilbert space H and for every bounded operator $T \colon H \to H$ having $\inf_{j\in J} \|Tx_j\| > 0$ there exists a partition $J = \bigcup_{i=1}^r J_i$ such that all restrictions $T|H_{J_i}$, $i = 1, \ldots, r$, are left invertible; here $H_{J'} = \operatorname{span}\{x_j \colon j \in J'\}$ for every $J' \subset J$. The conjecture is still open (June 2010). A stronger form (which is disproved in Section 3) claims the same property but for all summation basic sequences $\{x_j\}$.

2. Algebras $H^\infty|\sigma$ with a given constant δ_1

We start with some geometrical considerations. In this section the symbol b_λ always means the Blaschke factor with the zero λ in the upper half-plane, i. e.,

$$b_\lambda(z) = \frac{z-\lambda}{z-\overline{\lambda}} \cdot \frac{|1+\lambda^2|}{1+\lambda^2}.$$

Lemma 3. *The rectangle*

$$\left\{z\colon \frac{\sqrt{1+\varepsilon^2}}{\sqrt{1+\varepsilon^2}+\sqrt{2}\,\varepsilon} \le \frac{\operatorname{Im} z}{\operatorname{Im}\lambda} \le \frac{\sqrt{1+\varepsilon^2}}{\sqrt{1+\varepsilon^2}-\sqrt{2}\,\varepsilon},\ \frac{|\operatorname{Re}(z-\lambda)|}{\operatorname{Im}\lambda} \le \frac{\sqrt{2}\,\varepsilon}{\sqrt{1-\varepsilon^2}}\right\}$$

is inscribed into the circle $\{z\colon |b_\lambda(z)| \le \varepsilon\}$.

Proof. Put

$$a = \frac{\operatorname{Re}(z-\lambda)}{\operatorname{Im}\lambda} \qquad \text{and} \qquad b = \frac{\operatorname{Im} z}{\operatorname{Im}\lambda},$$

then

$$|b_\lambda(z)|^2 = \left|\frac{z-\lambda}{z-\bar\lambda}\right|^2 = \frac{a^2+(b-1)^2}{a^2+(b+1)^2}.$$

We have to check that the vertices of the rectangle are on the mentioned circle, i. e., we need to check that the equality

$$\frac{a^2+(b-1)^2}{a^2+(b+1)^2} = \varepsilon^2$$

holds if

$$a = \pm\frac{\sqrt{2}\,\varepsilon}{\sqrt{1-\varepsilon^2}}, \qquad b = \frac{\sqrt{1+\varepsilon^2}}{\sqrt{1+\varepsilon^2}\pm\sqrt{2}\,\varepsilon}.$$

We shall verify the required identity in the form $(a^2+b^2+1)(1-\varepsilon^2) = 2b(1+\varepsilon^2)$:

$$\begin{aligned}(a^2+b^2+1)(1-\varepsilon^2) &= \left(\frac{2\varepsilon^2}{1-\varepsilon^2} + \frac{1+\varepsilon^2}{(\sqrt{1+\varepsilon^2}\pm\sqrt{2}\,\varepsilon)^2} + 1\right)(1-\varepsilon^2) = \\ &2\varepsilon^2 + (1+\varepsilon^2)\frac{\sqrt{1+\varepsilon^2}\mp\sqrt{2}\,\varepsilon}{\sqrt{1+\varepsilon^2}\pm\sqrt{2}\,\varepsilon} + 1 - \varepsilon^2 = \\ &(1+\varepsilon^2)\frac{2\sqrt{1+\varepsilon^2}}{\sqrt{1+\varepsilon^2}\pm\sqrt{2}\,\varepsilon} = 2b(1+\varepsilon^2).\end{aligned}$$

□

Now, we are using Frostman shifts of an inner function Θ:

$$\Theta_c \stackrel{\text{def}}{=} \frac{\Theta + c}{1+\bar c\Theta}\,,$$

which is known to be a Blaschke product for almost all values c, $|c|<1$. In some cases it is easy to check that this is a Blaschke product *for all* $c \ne 0$. For example, this is the case for $\Theta = e^{iaz}$, $a>0$. Indeed, the inner function Θ_c is analytic in a neighborhood of any real point, therefore it could have a singular factor with a mass at infinity only. But there is no such factor because $\lim_{y\to+\infty}\Theta_c(iy) = c \ne 0$. For more details, see, for example, [Gar] or [Nik1].

Lemma 4. *Let z_k be zeroes of the Blaschke product*

$$B_{\alpha,\gamma} = \frac{e^{\pi i\gamma z} + e^{-\pi\alpha}}{1 + e^{\pi(i\gamma z-\alpha)}} = \prod_{k=-\infty}^{\infty} b_{z_k}\,,$$

i. e., $z_k = (2k+1+i\alpha)/\gamma$, $z_k \in \mathbb{Z}$. Then the strip

$$S_{\alpha,\gamma} = \left\{ z\colon \frac{\alpha\sqrt{1+\alpha^2}}{\sqrt{1+\alpha^2}+1} \le \gamma \operatorname{Im} z \le \frac{\alpha\sqrt{1+\alpha^2}}{\sqrt{1+\alpha^2}-1} \right\}$$

is in the set

$$\bigcup_{k=-\infty}^{\infty} \{z\colon |b_{z_k}(z)| < \varepsilon\},$$

if $\varepsilon > 1/\sqrt{1+2\alpha^2}$.

Proof. Apply Lemma 3 with $\lambda = z_k$ and $\varepsilon = 1/\sqrt{1+2\alpha^2}$. Then the sides of the rectangle are

$$\frac{\sqrt{1+\varepsilon^2}}{\sqrt{1+\varepsilon^2} \pm \sqrt{2}\,\varepsilon} = \frac{\sqrt{1+\alpha^2}}{\sqrt{1+\alpha^2}\pm 1} \qquad \text{and} \qquad \frac{\sqrt{2}\,\varepsilon}{\sqrt{1-\varepsilon^2}} = \frac{1}{\alpha},$$

i. e., the rectangle from Lemma 3 is

$$\left\{ z\colon \frac{\sqrt{1+\alpha^2}}{\sqrt{1+\alpha^2}+1} \le \frac{\gamma}{\alpha} \operatorname{Im} z \le \frac{\sqrt{1+\alpha^2}}{\sqrt{1+\alpha^2}-1},\ |\operatorname{Re}(z-z_k)| \le \frac{1}{\gamma} \right\}.$$

It is clear that the union of these rectangles gives just the required strip. □

Remark 1. Let us note that the set

$$S_{\alpha,\gamma} \setminus \bigcup_{k=-\infty}^{\infty} \left\{ z\colon |b_{z_k}(z)| < \frac{1}{\sqrt{1+2\alpha^2}} \right\}$$

consists of a discrete set of points

$$\frac{1}{\gamma}\left(2m + \frac{\alpha\sqrt{1+\alpha^2}}{\sqrt{1+\alpha^2}\pm 1}\, i \right)$$

on the upper and lower boundaries of the strip $S_{\alpha,\gamma}$ and the distance from any such point to the set of zeroes $\{z_k\}$ is equal to $1/\sqrt{1+2\alpha^2}$.

Lemma 5. *The Blaschke product $B = \prod_{n=0}^{\infty} B_{\alpha,\beta^n\rho}$ converges for all α, β, ρ such that $\alpha > 0$, $0 < \beta < 1$, $\rho > 0$. If*

$$\beta = \frac{\sqrt{1+\alpha^2}-1}{\sqrt{1+\alpha^2}+1}\,,$$

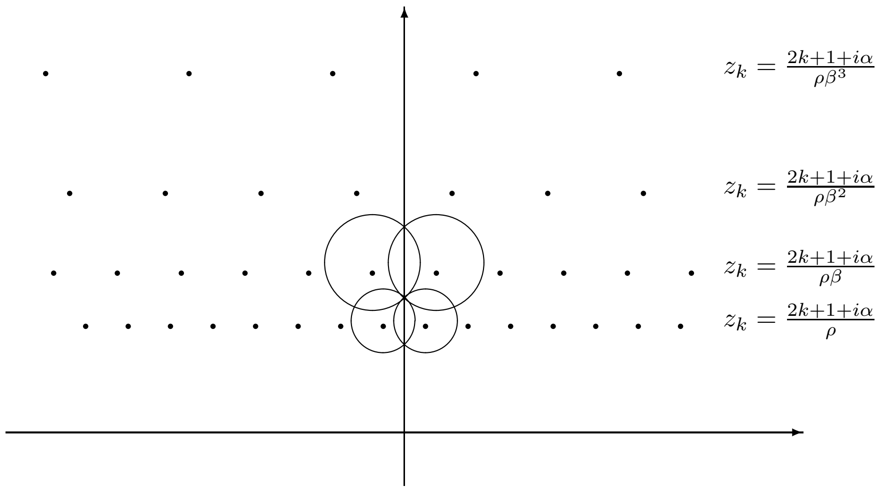

FIGURE 1.

then the half-plane

$$\Pi_{\alpha,\rho} = \left\{ z\colon\ \operatorname{Im} z \geq \frac{\alpha\sqrt{1+\alpha^2}}{\rho(\sqrt{1+\alpha^2}+1)} \right\}$$

is in the set

$$\bigcup_{\lambda\in\sigma(B)} \{z\colon |b_\lambda(z)| < \varepsilon\}$$

for $\varepsilon > 1/\sqrt{1+2\alpha^2}$.

(Fig. 1 illustrates zeroes of B and four circles $|b_\lambda(z)| = \frac{1}{\sqrt{1+2\alpha^2}}$ for zeroes $\lambda = \frac{\pm 1+i\alpha}{\rho}$ and $\lambda = \frac{\pm 1+i\alpha}{\beta\rho}$)

Proof. The following estimate implies convergence of B:

$$\begin{aligned} 1 - B_{\alpha,\beta^n\rho}(i) =& 1 - \frac{e^{-\pi\beta^n\rho} + e^{-\pi\alpha}}{1+e^{-\pi(\alpha+\beta^n\rho)}} = \\ & \frac{(1-e^{-\pi\beta^n\rho})(1-e^{-\pi\alpha})}{1+e^{-\pi(\alpha+\beta^n\rho)}} \leq \\ & (1-e^{-\pi\alpha})\pi\beta^n\rho. \end{aligned}$$

It remains to note that for $\beta = (\sqrt{1+\alpha^2}-1)/(\sqrt{1+\alpha^2}+1)$ and $\gamma_n = \beta^n\rho$ the upper boundary of the strip from Lemma 4 for $\gamma = \gamma_{n-1}$ coincides with the lower boundary of the strip for $\gamma = \gamma_n$. Therefore the union of these strips gives just the required half-plane. $\square$

Remark 2. The set of points on the imaginary axis

$$v_n = \frac{1}{\rho\beta^n}\frac{\alpha\sqrt{1+\alpha^2}}{\sqrt{1+\alpha^2}+1}\, i$$

(the points of intersection of four corresponding circles as on Fig.1) is included into

$$\Pi_{\alpha,\rho} \setminus \bigcup_{\lambda\in\sigma(B)}^{\infty} \left\{z\colon |b_\lambda(z)| < \frac{1}{\sqrt{1+2\alpha^2}}\right\}.$$

Every point v_n has four nearest zeroes of B, namely, $(i\alpha\pm 1)/(\rho\beta^n)$ and $(i\alpha\pm 1)/(\rho\beta^{n-1})$ with the pseudohyperbolic distance just $1/\sqrt{1+2\alpha^2}$ from each of them.

Recall that pseudohyperbolic distance between two points $z, w \in \mathbb{C}_+$ is defined by

$$|b_w(z)| = \left|\frac{z-w}{z-\bar{w}}\right|$$

and between two points $z, w \in \mathbb{D}$:

$$|b_w(z)| = \left|\frac{z-w}{1-\bar{w}z}\right|.$$

Lemma 6. *For the Blaschke product $B = \prod_{n=0}^\infty B_{\alpha,\beta^n\rho}$ a lower estimate*

$$|B(x+iy)| \geq \exp\left\{-\frac{\left(1+e^{-\pi\alpha}\right)\pi\rho y}{\left(e^{-\pi\rho y}-e^{-\pi\alpha}\right)(1-\beta)}\right\}$$

is true in the strip $0 < y < \frac{\alpha}{\rho}$. In the complementary half-plane $y > \frac{\alpha}{\rho}$ we have the following upper estimate

$$|B(x+iy)| \leq \exp\left\{-\frac{\log(\cosh\pi\alpha)\cdot\log\frac{\rho y}{\alpha}}{\log\frac{1}{\beta}}\right\}.$$

Proof. For the product $B_{\alpha,\gamma}$ we have

$$|B_{\alpha,\gamma}(x+iy)|^2 = \frac{e^{-2\pi\gamma y}+e^{-2\pi\alpha}+2e^{-\pi(\gamma y+\alpha)}\cos\pi\gamma x}{1+e^{-2\pi(\gamma y+\alpha)}+2e^{-\pi(\gamma y+\alpha)}\cos\pi\gamma x}$$

and therefore

$$\left|\frac{e^{-\pi\gamma y}-e^{-\pi\alpha}}{1-e^{-\pi(\gamma y+\alpha)}}\right| \leq |B_{\alpha,\gamma}(x+iy)| \leq \frac{e^{-\pi\gamma y}+e^{-\pi\alpha}}{1+e^{-\pi(\gamma y+\alpha)}}.$$

Now, we deduce an estimate from below assuming $0 < y < \frac{\alpha}{\rho}$:

$$\begin{aligned}
\log\frac{1}{|B(x+iy)|} &= \sum_{n=0}^{\infty}\log\frac{1}{|B_{\alpha,\beta^n\rho}(x+iy)|} \le \sum_{n=0}^{\infty}\log\frac{1-e^{-\pi(\beta^n\rho y+\alpha)}}{e^{-\pi\beta^n\rho y}-e^{-\pi\alpha}}\\
&\le \sum_{n=0}^{\infty}\left(\frac{1-e^{-\pi(\beta^n\rho y+\alpha)}}{e^{-\pi\beta^n\rho y}-e^{-\pi\alpha}}-1\right) = \sum_{n=0}^{\infty}\frac{\left(1+e^{-\pi\alpha}\right)\left(1-e^{-\pi\beta^n\rho y}\right)}{e^{-\pi\beta^n\rho y}-e^{-\pi\alpha}}\\
&\le \left(1+e^{-\pi\alpha}\right)\sum_{n=0}^{\infty}\frac{\pi\beta^n\rho y}{e^{-\pi\rho y}-e^{-\pi\alpha}} = \frac{\left(1+e^{-\pi\alpha}\right)\pi\rho y}{\left(e^{-\pi\rho y}-e^{-\pi\alpha}\right)(1-\beta)},
\end{aligned}$$

as it was claimed. To estimate $|B(x+iy)|$ from above we replace B by a finite product $\prod_{0\le n\le N} B_{\alpha,\beta^n\rho}$, where

$$N \stackrel{\text{def}}{=} \frac{\log\frac{\rho y}{\alpha}}{\log\frac{1}{\beta}}.$$

The number of such indices n is $[N]+1 > N$. Since for these n we have

$$\beta^n\rho y \ge \alpha,$$

for each factor we get an estimate

$$|B_{\alpha,\beta^n\rho}(x+iy)| \le \frac{e^{-\pi\beta^n\rho y}+e^{-\pi\alpha}}{1+e^{-\pi(\beta^n\rho y+\alpha)}} \le \frac{2e^{-\pi\alpha}}{1+e^{-2\pi\alpha}} = \frac{1}{\cosh\pi\alpha}.$$

Therefore for the whole product we have

$$|B(x+iy)| \le (\cosh\pi\alpha)^{-N}.$$

□

From now on, we fix

$$\beta = \frac{\sqrt{1+\alpha^2}-1}{\sqrt{1+\alpha^2}+1},$$

and consider $B = \prod_{n=0}^{\infty} B_{\alpha,\beta^n\rho}$ corresponding to this β.

Theorem 7.

$$\delta_1(H^\infty/BH^\infty) = \frac{1}{\sqrt{1+2\alpha^2}}.$$

Moreover, there exists an absolute constant c and another constant $C = C(\delta_1)$ such that

$$c_1(\delta) \le \max\left\{\frac{c}{(\delta-\delta_1)^2}\log\frac{1}{\delta-\delta_1},\ C\right\}$$

for every δ, $\delta_1 < \delta \le 1$.

The proof of the Theorem is contained in two following lemmata, where δ_1 means simply the number $\frac{1}{\sqrt{1+2\alpha^2}}$. After proving these lemmata we can conclude that $\delta_1 = \delta_1(H^\infty/BH^\infty)$.

Lemma 8. *Let $\delta > \delta_1$. Then*

$$c_1(\delta) \le \max\Big\{\frac{c}{(\delta-\delta_1)^2}\log\frac{1}{\delta-\delta_1},\ C\Big\}$$

for some an absolute constant c and another constant $C = C(\delta_1)$.

Proof. First we check that the function $|f(z)|+|B(z)|$ can be separated from zero by some constant η depending on α and δ only. By Lemma 6 in the strip

$$0 < y\rho \le \frac{\alpha\sqrt{1+\alpha^2}}{\sqrt{1+\alpha^2}+1}$$

we have the estimate

$$|B(x+iy)| \ge \exp\left\{-\frac{\alpha\sqrt{1+\alpha^2}\,(e^{\pi\alpha}+1)}{2\left(e^{\frac{\pi\alpha}{\sqrt{1+\alpha^2}+1}}+1\right)}\right\}.$$

Now we check that $|f(z)|$ is separated from zero in the half-plane

$$y\rho \ge \frac{\alpha\sqrt{1+\alpha^2}}{\sqrt{1+\alpha^2}+1}.$$

Fix any ε, $\delta > \varepsilon > \delta_1$. By Lemma 5

$$\left\{z\colon\ \operatorname{Im} z \ge \frac{\alpha\sqrt{1+\alpha^2}}{\rho\big(\sqrt{1+\alpha^2}+1\big)}\right\} \subset \bigcup_{\lambda\in\sigma(B)} \{z\colon |b_\lambda(z)| < \varepsilon\},$$

and therefore it is enough to verify that f is separated from zero on each disk $\{z\colon |b_\lambda(z)| < \varepsilon\}$, $\lambda \in \sigma(B)$, uniformly with respect to λ.

By the Schwarz lemma we have

$$\left|\frac{f(z)-f(\lambda)}{1-\overline{f(\lambda)}f(z)}\right| \le |b_\lambda(z)|,$$

i. e., for all point z of the disk $\{z\colon |b_\lambda(z)| < \varepsilon\}$ we have

$$\left|\frac{f(z)-f(\lambda)}{1-\overline{f(\lambda)}f(z)}\right| < \varepsilon.$$

Rewriting the inequality

$$\left|\frac{a+b}{1+\bar{a}b}\right| \le \frac{|a|+|b|}{1+|a|\,|b|}$$

(which is, in fact, the triangle inequality for the hyperbolic metric for the points a, b, and 0) in the form

$$|b| \geq \frac{|a| - \left|\frac{a+b}{1+\bar{a}b}\right|}{1 - |a|\left|\frac{a+b}{1+\bar{a}b}\right|}$$

with $a = f(\lambda)$ and $b = -f(z)$ we get

$$|f(z)| \geq \frac{\delta - \varepsilon}{1 - \delta\varepsilon} > \frac{\delta - \delta_1}{1 - \delta\delta_1}\,.$$

Therefore, in the whole half-plane we have

$$|f(z)| + |B(z)| \geq \eta\,,$$

where

$$\eta = \min\left\{\exp\left[-\frac{\alpha\sqrt{1+\alpha^2}\,(e^{\pi\alpha}+1)}{2\Big(e^{\frac{\pi\alpha}{\sqrt{1+\alpha^2}+1}}+1\Big)}\right],\ \frac{\delta-\delta_1}{1-\delta\delta_1}\right\}\,. \tag{2.1}$$

Finally, by the Carleson corona theorem (see, e.g. [Nik2]), we know that there exists a solution h of the Bezout equation $fh + Bg = 1$ with a norm estimate

$$\|h\|_\infty \leq \frac{c}{\eta^2}\log\frac{1}{\eta}\,,$$

which means that

$$c_1(\delta) \leq \frac{c}{\eta^2}\log\frac{1}{\eta}\,.$$

Recall that $\delta_1 = \frac{1}{\sqrt{1+2\alpha^2}}$. If the first term in (2.1) is less than the second one, we have

$$\eta = \eta(\delta_1) = \exp\left[-\frac{\alpha\sqrt{1+\alpha^2}\,(e^{\pi\alpha}+1)}{2\Big(e^{\frac{\pi\alpha}{\sqrt{1+\alpha^2}+1}}+1\Big)}\right],$$

and we can put

$$C(\delta_1) = \frac{c}{\eta^2(\delta_1)}\log\frac{1}{\eta(\delta_1)}\,.$$

If the second term is smaller, we have

$$c_1(\delta) \leq \frac{c}{(\delta-\delta_1)^2}\log\frac{1}{\delta-\delta_1}\,.$$

$\square$

Lemma 9. *Let $\delta \leq \delta_1$. Then $c_1(\delta) = +\infty$.*

Proof. Consider a sequence of points v_n from Remark 2 and put $f_n = b_{v_n}$. As it was mentioned in Remark 2,

$$|b_{v_n}(\lambda)| \geq \frac{1}{\sqrt{1+2\alpha^2}} = \delta_1 \geq \delta \qquad \forall \lambda \in \sigma(B)\,.$$

We would like to estimate from below the H^∞ norm of a solution g_n of the Bezout equation $g_n f_n + B h_n = 1$. Since

$$\|g_n\|_\infty = \|1 - B h_n\|_\infty = \|h_n - \bar{B}\|_\infty \geq \|h\|_\infty - 1$$

and

$$\|h_n\|_\infty \geq |h_n(v_n)| = \frac{1}{|B(v_n)|}\,,$$

by the estimate of Lemma 6 we obtain

$$\|g_n\|_\infty \to \infty\,,$$

what yields $c_1(\delta) = +\infty$. □

Remark 3. Taking an arbitrary δ, $\delta < \delta_1$, and using the above construction, it is easy to construct a function f with the properties $\|f\|_\infty \leq 1$, $|f(\lambda)| \geq \delta$ for every $\lambda \in \sigma$, which is not invertible in H^∞/BH^∞, so that there is no bounded solution g, h to the Bezout equation $gf + Bh = 1$. Indeed, it is sufficient to take for f a product of the factors b_{v_n} with sufficiently rare subsequence of zeroes v_n to ensure the condition $|f(\lambda)| \geq \delta$. However, for the Blaschke product B from Theorem 7, we do not know whether there exists such a function in the case $\delta = \delta_1$. In order to guarantee this property, i. e., to have a noninvertible element f of the algebra H^∞/BH^∞ with $\delta_1 \leq f(\lambda) \leq \|f\|_\infty \leq 1$ $(\lambda \in \sigma(B))$, we need a Blaschke product B with more sophisticated zero set, which will be exhibited in the following theorem.

Theorem 10. *For an arbitrary fixed number δ_1 from $(0,1)$ there exists a Blaschke product B such that*
1) $c_1(\delta, H^\infty/BH^\infty) < \infty$ *for every* δ, $\delta_1 < \delta \leq 1$;
2) *there exists a function f satisfying $\delta_1 \leq |f(\lambda)| \leq \|f\|_\infty \leq 1$ for $\lambda \in \sigma(B)$, but $\frac{1}{f} \notin H^\infty/BH^\infty$.*

Proof. **Step 1.** We start with an arbitrary bounded increasing sequence of positive number α_n with $\alpha = \lim \alpha_n$, $\delta_1 \stackrel{\text{def}}{=} \frac{1}{\sqrt{1+2\alpha^2}}$. Our Blaschke product B will be of the form

$$B(z) = \prod_{n=1}^{\infty} \prod_{m=0}^{m_n - 1} B_{\alpha_n, \beta_n^m \rho_n}(z)\,,$$

where

$$\beta_n = \frac{\sqrt{1+\alpha_n^2}-1}{\sqrt{1+\alpha_n^2}+1}$$

and

$$\rho_{n+1} = \rho_n \beta_n^{m_n} \frac{\sqrt{1+\alpha_n^2}+1}{\alpha_n\sqrt{1+\alpha_n^2}} \cdot \frac{\alpha_{n+1}\sqrt{1+\alpha_{n+1}^2}}{\sqrt{1+\alpha_{n+1}^2}+1} \,.$$

The initial value $\rho_1 = \rho$ can be taken arbitrarily. The claimed noninvertible function f will be the following Blaschke product

$$f(z) = \prod_{n=0}^{\infty} b_{v_n}(z) \,,$$

where

$$v_n = \frac{1}{\rho_{n+1}} \cdot \frac{\alpha_{n+1}\sqrt{1+\alpha_{n+1}^2}}{\sqrt{1+\alpha_{n+1}^2}+1}\, i = \frac{1}{\rho_n \beta_n^{m_n-1}} \cdot \frac{\alpha_n\sqrt{1+\alpha_n^2}}{\sqrt{1+\alpha_n^2}-1}\, i \,,$$

i. e., we put the root v_n on the common boundary of the last strip defined by α_n and the first strip defined by α_{n+1}. So, the only parameters, which are in our disposition, are the numbers m_n of strips of equal hyperbolic width or, in other words, the distances between the neighbor roots v_n. We subordinate these distance to the following condition

$$\left|\frac{v_l - v_k}{v_l + v_k}\right| \geq \left(\delta\sqrt{1+2\alpha_{l+1}^2}\right)^{2^{-k}} \,. \tag{2.2}$$

If we take any zero λ of the Blaschke product B with $\operatorname{Im} v_{n-1} < \operatorname{Im}\lambda < \operatorname{Im} v_n$, then

$$\begin{aligned}
|f(\lambda)| &= \prod_{k=0}^{\infty} |b_{v_k}(\lambda)| = \prod_{k=0}^{n-2} |b_{v_k}(\lambda)| \cdot |b_{v_{n-1}}(\lambda) b_{v_n}(\lambda)| \cdot \prod_{k=n+1}^{\infty} |b_{v_k}(\lambda)| \\
&\geq \prod_{k=0}^{n-2} |b_{v_k}(v_{n-1})| \cdot |b_{v_{n-1}}(\lambda) b_{v_n}(\lambda)| \cdot \prod_{k=n+1}^{\infty} |b_{v_k}(v_n)| \\
&\geq \prod_{k=0}^{n-2} \left(\delta\sqrt{1+2\alpha_n^2}\right)^{2^{-k}} \cdot |b_{v_{n-1}}(\lambda) b_{v_n}(\lambda)| \cdot \prod_{k=n+1}^{\infty} \left(\delta\sqrt{1+2\alpha_{n+1}^2}\right)^{2^{-k}} \\
&\geq \left(\delta\sqrt{1+2\alpha_n^2}\right)^{1-3\cdot 2^{-n}} \cdot |b_{v_{n-1}}(\lambda) b_{v_n}(\lambda)| \,.
\end{aligned}$$

Thus, would we guarantee for any root λ in the strip between v_{n-1} and v_n the estimate

$$|b_{v_{n-1}}(\lambda)b_{v_n}(\lambda)| \geq \frac{\left(\delta\sqrt{1+2\alpha_n^2}\right)^{3\cdot 2^{-n}}}{\sqrt{1+2\alpha_n^2}}, \tag{2.3}$$

we will immediately obtain the required estimate for f: $|f(\lambda)| \geq \delta$.

Step 2. We shall construct the roots v_n by induction. Assume that all v_k for $k < n$ are already fixed and we need to choose v_n. First of all we have to take v_n far enough from the preceding roots in order to satisfy (2.2) for $k = n$ and all $l < n$ as well as for $l = n$ and all $k < n$.

Note that we need to check condition (2.3) only for the roots λ of B with positive real part and the nearest to the imaginary axis, i. e., for $\lambda = (1+i\alpha_n)/\rho_n\beta_n^m$, because the hyperbolic distance between all other λ with positive real part and any v_k is strictly larger, but the consideration for λ with negative real part can be omitted due to the symmetry.

Now, we would like to reduce the problem to the case of two roots of B only, the nearest roots to one of the zeroes of f, either v_{n-1} or v_n, i. e., for $m = 0$ and $m = m_n - 1$.

For the root $\lambda = (1+i\alpha_n)/\rho_n$, we have

$$|b_{v_{n-1}}(\lambda)| = \frac{1}{\sqrt{1+2\alpha_n^2}},$$

and hence (2.3) turns into

$$|b_{v_n}(\lambda)| \geq \left(\delta\sqrt{1+2\alpha_n^2}\right)^{3\cdot 2^{-n}}. \tag{2.4}$$

For the root $\lambda = (1+i\alpha_n)/\rho_n\beta_n^{m_n-1}$ we have

$$|b_{v_n}(\lambda)| = \frac{1}{\sqrt{1+2\alpha_n^2}},$$

therefore (2.3) turns into

$$|b_{v_{n-1}}(\lambda)| \geq \left(\delta\sqrt{1+2\alpha_n^2}\right)^{3\cdot 2^{-n}}. \tag{2.5}$$

In fact, both (2.4) and (2.5) follow from (2.2), however we do not want to enter into these additional estimations and simply add (2.4)–(2.5) to the list of requirements for the inductive choice of m_n. Now, we check that (2.4)–(2.5) are fulfilled, as well as (2.2), for m_n sufficiently large.

Let us consider the behavior of the function $\phi_a(t)$,

$$\phi_a(t) \stackrel{\text{def}}{=} |b_{ia}\big((1+i\alpha_n)t\big)|^2 = \frac{(a-\alpha_n t)^2 + t^2}{(a+\alpha_n t)^2 + t^2}.$$

Since

$$\frac{\phi_a'(t)}{\phi_a(t)} = \frac{4\alpha_n a[t^2(1+\alpha_n^2)-a^2]}{[t^2(1+\alpha_n^2)+a^2]^2-4\alpha_n^2a^2t^2}\,,$$

the function ϕ_a monotonously decreases from 1 to its minimal value, when t changes from 0 to $a/\sqrt{1+\alpha_n^2}$, and then increases tending again to 1 as $t\to\infty$. If we consider the product of two such functions $\phi_a(t)\phi_b(t)$ with sufficiently large b/a, then it is clear that this product has two local minima: first of them tends to $a/\sqrt{1+\alpha_n^2}$ monotonously decreasing as $b\to\infty$, and the second one tends to $b/\sqrt{1+\alpha_n^2}$ monotonously increasing as $a\to 0$.

We apply these arguments to our requirement (2.3). To this end, we set

$$a=|v_{n-1}|=\frac{\alpha_n\sqrt{1+\alpha_n^2}}{\rho_n(\sqrt{1+\alpha_n^2}+1)}\,,\qquad b=|v_n|=\frac{\alpha_n\sqrt{1+\alpha_n^2}}{\rho_n\beta_n^{m_n}(\sqrt{1+\alpha_n^2}+1)}\,,$$

and

$$\lambda=\frac{1+i\alpha_n}{\rho_n\beta_n^m}\,,$$

and will compare the values of our function at the points $t=t_m=1/\rho_n\beta_n^m$, $m=0,\dots,m_n-1$, in order to guarantee that the minimal value is attained either for $m=0$ or for $m=m_n-1$, where by our assumption either (2.4) or (2.5) is fulfilled. To this aim, we note that

$$t_0=\frac{1}{\rho_n}>\frac{\alpha_n}{\rho_n(\sqrt{1+\alpha_n^2}+1)}=\frac{a}{\sqrt{1+\alpha_n^2}}\,,$$

and therefore, for m_n sufficiently large, the point of the minimum is less then t_0 and the function $\phi_a(t)\phi_b(t)$ is increasing at t_0. Symmetrically,

$$t_{m_n-1}=\frac{1}{\rho_n\beta_n^{m_n-1}}<\frac{\alpha_n}{\rho_n\beta_n^{m_n-1}(\sqrt{1+\alpha_n^2}-1)}=\frac{b}{\sqrt{1+\alpha_n^2}}\,,$$

and therefore, for m_n sufficiently large, the point of the minimum is bigger then t_{m_n-1} and the function $\phi_a(t)\phi_b(t)$ is decreasing at t_{m_n-1}. It follows that conditions (2.3) is fulfilled for m_n large enough.

Thus, we have proved that conditions (2.2) and (2.3) are fulfilled if the sequence m_n increases fast enough. Therefore, the construction of the Blaschke product B and a function f such that $\|f\|_\infty=1$, $|f(\lambda)|\ge\delta_1$ for all λ, $\lambda\in\sigma(B)$, is completed. The function f represents a noninvertible element of H^∞/BH^∞. Indeed, if we assume that f is invertible in H^∞/BH^∞, i. e., there exist two H^∞-functions g and h such that $fg+Bh=1$, then we come to a contradiction, because

$$\lim_{n\to\infty}\big(f(v_n)g(v_n)+B(v_n)h(v_n)\big)=0\,.$$

So, we have finished the proof of the second statement of the Theorem.

Step 3. In order to complete the proof, we need to check that any other function f satisfying conditions $\|f\|_\infty = 1$, $|f(\lambda)| \ge \delta$ for all λ, $\lambda \in \sigma(B)$, and for arbitrary δ, $\delta > \delta_1$, represents an invertible element of H^∞/BH^∞.

Fix such a δ and such a function f. We need to check that

$$\inf_{\operatorname{Im} z>0}(|f(z)| + |B(z)|) > 0. \tag{2.6}$$

As in the proof of Theorem 7, we take an arbitrary ε, $\delta_1 < \varepsilon < \delta$, and split the upper half-plane in two parts:

$$\Pi_\varepsilon \stackrel{\text{def}}{=} \cup_{\lambda\in\sigma(B)}\{z\colon |b_\lambda(z)| \le \varepsilon\}$$

and its complement Π_ε^c. We will check that f is bounded away from zero on Π_ε and B does it on Π_ε^c.

If $|b_\lambda(z)| \le \varepsilon$, then by Schwarz' lemma

$$\left|\frac{f(z) - f(\lambda)}{1 - \overline{f(\lambda)}f(z)}\right| \le |b_\lambda(z)| \le \varepsilon\,,$$

and again, as in the proof of Lemma 8, using the triangle inequality in the form

$$|b| \ge \frac{|a| - \left|\frac{a+b}{1+\bar a b}\right|}{1 - |a|\left|\frac{a+b}{1+\bar a b}\right|}$$

with $a = f(\lambda)$ and $b = -f(z)$ we get

$$|f(z)| \ge \frac{\delta - \varepsilon}{1 - \delta\varepsilon}$$

for arbitrary z from Π_ε.

On the complement Π_ε^c, we estimate $|B(z)|$ splitting the product B into two subproducts $B = B'B''$. Namely, we fix a number N so that $\delta\sqrt{1 + 2\alpha_n^2} > 1$ for $n \ge N$ and put

$$B' = \prod_{n=1}^{N-1}\prod_{m=0}^{m_n-1} B_{\alpha_n,\beta_n^m\rho_n}\,, \qquad B'' = \prod_{n=N}^{\infty}\prod_{m=0}^{m_n-1} B_{\alpha_n,\beta_n^m\rho_n}\,.$$

Note that the first product is an interpolating Blaschke product. Indeed, all $B_{\alpha,\gamma}$ are interpolating, because due to the relation

$$\Bigl(\frac{B}{b_\lambda}\Bigr)(\lambda) = 2i\cdot\operatorname{Im}\lambda\cdot\frac{dB}{dz}(\lambda)$$

we have

$$B_k(z_k) = \frac{\pi\alpha}{\sinh\pi\alpha}\,,$$

where $B_k = B_{\alpha,\gamma}/b_{z_k}$, $z_k = (2k+1+i\alpha)/\gamma$. Therefore zeroes of B' form a finite union of interpolating sets. Since they are uniformly separated, the whole product B' is interpolating as well. Using a generalized form of the Carleson condition (see, e.g., [Nik1]–[Nik2])

$$|B'(z)| \geq c \inf_{\lambda\in\sigma(B')} |b_\lambda(z)|\,,$$

we get $|B'(z)| \geq c\varepsilon$ in Π^c_ε. As to the second product B'', we can use the estimate of Lemma 6 for $\rho = \rho_N$. The estimate of Lemma 6 was obtained for the strips of equal hyperbolic width, but in our situation the width of the strips decreases, because α_n is increasing. This means that the hyperbolic distance from any point below the first strip to the corresponding zero of $B_{\alpha_n,\gamma_{n,m}}$ is strictly bigger than that distance in the case when all α_n are equal to α. Therefore, below the first strip, each factor $|B_{\alpha_n,\gamma_{n,m}}|$ is strictly larger than in the equidistant case. The whole half-plane

$$\operatorname{Im} z > \operatorname{Im} v_{N-1} = \frac{1}{\rho_N}\cdot\frac{\alpha_N\sqrt{1+\alpha_N^2}}{\sqrt{1+\alpha_N^2}+1}$$

is in the set Π_ε by Lemma 5, therefore B'' is separated from zero in the set Π^c_ε, whence the whole B is separated from zero on Π^c_ε. So, condition (2.6) is fulfilled what means the invertibility of f in the algebra H^∞/BH^∞, i. e., $c(\delta) < \infty$ for any δ, $\delta > 1/\sqrt{1+2\alpha^2}$. □

3. A version of the restricted invertibility conjecture

3.1. **Bourgain–Tzafriri's restricted invertibility theorem.** The following statement is known as *Bourgain–Tzafriri's restricted invertibility theorem.*

Theorem 11. ([BTz]) *Whatever are a bounded operator T on a Hilbert space H and an orthogonal basis $\{e_j\}_{j\in\mathbb{N}}$ satisfying* $\inf_j \frac{\|Te_j\|}{\|e_j\|} > 0$, *there exists a subset $I \subset \mathbb{N}$ of positive upper density*

$$0 < \overline{\operatorname{dens}}(I) \stackrel{\text{def}}{=} \limsup_{n\to\infty} \frac{|I\cap\{1,2,\dots,n\}|}{n}$$

such that the restriction $T|H_I$ is left invertible:

$$\inf\{\|Tx\|\colon x\in H_I, \|x\| = 1\} > 0\,,$$

where $H_I = \operatorname{span}\{e_j\colon j\in I\}$.

See also [SS] for a generalization and a simpler proof of a matrix version of Bourgain–Tzafriri's Theorem.

The following conjecture often is quoted as Bourgain–Tzafriri's *restricted invertibility conjecture* (**RIC**) (it seems although that these

authors never actually stated this as a conjecture). It is known that the famous Kadison–Singer conjecture on pure states on C^*-algebras (see [KS], [CT]) implies **RIC**: it is proved in [CCLV] that if Kadison–Singer problem has a positive solution then the **RIC** has as well. For more details about these conjectures we refer to the papers mentioned above, as well as to a WEB page [ARCC]. Both conjectures are still open (June 2010).

3.2. **Restricted Invertibility Conjecture (RIC).**

Conjecture. *For every bounded operator T on a Hilbert space H and every orthogonal basis $\{e_j\}_{j\in\mathbb{N}}$ satisfying $\inf_j \frac{\|Te_j\|}{\|e_j\|} > 0$, there exists a finite partition $\bigcup_{s=1}^r I_s = \mathbb{N}$ such that all restrictions $T|H_{I_s}$ are left invertible.*

It is easy to see that the **RIC** is equivalent to require the same quality partitions for every bounded T and every *unconditional* basis in H (in place of orthogonal ones). Knowing no much progress in this conjecture during the last 20 years, we can try to approach the truth treating first some stronger conjectures. Namely, we can replace here an "unconditional basis" by a "Schauder basis", and even by a "summation basis". *We denote the corresponding conjectures by* **B-RIC** *and* **SB-RIC**, *respectively.*

Precisely, a *summation basis* relative to a (triangular) matrix $V = \{v_{nj}\}$ of scalars v_{nj} is a sequence $\{e_j\}_{j\in\mathbb{N}}$ in H such that for every $x \in H$ there exists a unique sequence of scalars $\{a_j\}_{j\in\mathbb{N}}$ satisfying $x = (V)\sum_{j\geq 1} a_j e_j$, which means the following:

- $v_{nj} = 0$ for $j > n$;
- $x = \lim_{n\to\infty} \sum_{j=1}^n v_{nj} a_j e_j = x$ (norm convergence).

Clearly, **SB-RIC** $\implies$ **B-RIC** $\implies$ **RIC**. Here, we present a counterexample to the **SB-RIC**.

3.3. **Counterexample.**

Theorem 12. *Given δ, $0 < \delta < 1$, there exists a sequence $\{e_j\}_{j\in\mathbb{N}}$ in a Hilbert space H satisfying the following properties.*

(1) *$\{e_j\}_{j\in\mathbb{N}}$ is a summation basis (relative to a triangular matrix).*
(2) *$\{e_j\}_{j\in\mathbb{N}}$ is block orthogonal: there exists an increasing sequence of integers n_s such that $H_{[n_s,n_{s+1})} \perp H_{[n_t,n_{t+1})}$ for every $s \neq t$, where $H_{[n_s,n_{s+1})} = \operatorname{span}\{e_j : n_s \leq j < n_{s+1}\}$.*
(3) *There exists a bounded operator $A\colon H \to H$ satisfying $\|A\| \leq 1$, $Ae_j = \lambda_j(A)e_j$, $\delta \leq |\lambda_j(A)| = \frac{\|Ae_j\|}{\|e_j\|} \leq \|A\| \leq 1$ $(j \in \mathbb{N})$,*

and such that for every finite partition $\bigcup_{s=1}^{r} I_s = \mathbb{N}$ there is a restriction $A|H_{I_s}$ $(1 \le s \le r)$, which is NOT left invertible.

(4) *Every bounded operator $T\colon H \to H$ satisfying $Te_j = \lambda_j(T)e_j$ $(j \in \mathbb{N})$ and $1 \ge \|T\| \ge \inf_j |\lambda_j(T)| > \delta$ is invertible.*

Proof. We use our main construction from Theorem 10 replacing the upper half-plane by the unit disk. Namely, given δ (δ_1 in the Theorem), $0 < \delta < 1$, there exists a Blaschke sequence $\sigma = \{z_j\}$ of distinct points in $\mathbb{D}$ such that

a) for every $f \in H^\infty$ with $\delta < \inf_j |f(z_j)|$ and $\|f\|_\infty \le 1$, we have $\frac{1}{f} \in H^\infty|\sigma$;

b) there is an H^∞ function g such that $\delta \le |g(z_j)| \le \|g\|_\infty \le 1$ but $\frac{1}{g} \notin H^\infty|\sigma$.

Now, we interpret a) and b) in terms of the model operator M_B^* and the reproducing kernels $x_j = \frac{(1-|z_j|^2)^{1/2}}{1-\bar{z}_j z}$.

First, by (a scalar version of) the commutant lifting theorem, an operator T from point (4) of the Theorem is of the form $T = f(M_B)^*$, where f satisfies all properties from a): $f \in H^\infty$, $\delta < \inf_j |f(z_j)|$ and $\|f\|_\infty \le 1$. Hence, $\frac{1}{f} \in H^\infty|\sigma$, which means that T is invertible (and proves point (4)).

Secondly, in order to fix statements (1)–(3), we restate item b) above in terms of the same model operator. Namely, for an operator

$$T = g(M_B)^*$$

with a function g from b) we have $\|T\| \le 1$, $Tx_j = \lambda_j(T)x_j$, $\delta \le |\lambda_j(T)| = |g(z_j)| \le \|T\| \le 1$ $(j \in \mathbb{N})$, and $\inf\{\|Tx\|\colon x \in K_B, \|x\| = 1\} = 0$.

Notice that if we would like to restrict ourselves to properties (1) and (3) only, we simply set $A = T = f(M_B)^*$. Property (1) follows from the fact that the sequence $\{x_j\}_{j\in\mathbb{N}}$ corresponding to a Blaschke sequence $\{z_j\}_{j\in\mathbb{N}}$ is a summation basis, [Nik1], p. 194. In order to check (3), suppose that there exists a partition $\bigcup_{s=1}^{r} \sigma_s = \mathbb{N}$ such that all restrictions $T|H_{\sigma_s}$ are left invertible:

$$0 < \inf\{\|Tx\|\colon x \in H_{\sigma_s}, \|x\| = 1\}$$

for every s, $1 \le s \le r$. We lead this to contradiction as follows. Let B_s be the Blaschke product whose zero sequence is σ_s. Since the restriction $T|H_{\sigma_s} = g(M_{B_s})^*$ is, in fact, invertible, there exist functions $f_s, h_s \in H^\infty$ such that $gf_s + B_s h_s = 1$. Hence, $B \cdot \prod_{s=1}^{r} h_s = \prod_{s=1}^{r}(1 - gf_s) = 1 - gF$, where $F \in H^\infty$. This shows that the operator $T = g(M_B)^*$ is invertible, what contradicts the construction of T. Therefore,

a counterexample satisfying properties (1), (3), and (4) of the Theorem is constructed.

In order to satisfy property (2), we modify the previous construction in the following way. Let $N \in \mathbb{N}$ and $T_N = T|H_N$ be the restriction of T to

$$H_N = \operatorname{span}\{x_j \colon 1 \le j \le N\}\,.$$

Then

$$\|T_N\| \le 1, \quad T_N x_j = \lambda_j(T_N)x_j\,,$$
$$\delta \le |\lambda_j(T_N)| \le \|T\| \le 1 \quad (1 \le j \le N)\,,$$

and

$$\lim_{N\to\infty} \inf\{\|T_N x\| \colon x \in H_N, \|x\| = 1\} = 0\,.$$

Now, we set

$$A = \sum_{N\ge 1} \oplus T_N\,,$$

which is defined coordinate-wise on an (l^2) orthogonal sum

$$H = \sum_{N\ge 1} \oplus H_N\,.$$

In particular, this means that the point spectrum of A is $\{\lambda_j(T)\}_{j\ge 1}$ but each eigenvalues is repeated infinitely many times.

Next, we denote $\{e_j\}_{j\in\mathbb{N}}$ the sequence of eigenvectors of A ordered naturally: if $f_k = (\delta_{N,k})_{N\ge 1} \in l^2$, then

$$\{e_j\}_{j\in\mathbb{N}} = (x_1f_1, x_1f_2, x_2f_2, \ldots, x_1f_N, x_2f_N, \ldots, x_Nf_N, x_1f_{N+1}, \ldots)\,,$$

or, more formally,

$$e_j = x_m f_N\,, \qquad \text{where} \quad N = [\sqrt{2j} + \frac{1}{2}], \quad m = j - \frac{N(N-1)}{2}\,.$$

Show that $\{e_j\}_{j\in\mathbb{N}}$ satisfies properties (1) and (2), and A fulfils all requirements of (3).

Indeed, properties (1) and (2) for $\{e_j\}$ easily follow from the property (1) for $\{x_j\}$ and a block orthogonal nature of $\{e_j\}$.

In order to prove (3), suppose the contrary, i. e., that there exists a finite partition $\bigcup_{s=1}^r I_s = \mathbb{N}$ such that all restrictions $A|H_{I_s}$ $(1 \le s \le r)$ are left invertible. Taking an intersection of $\bigcup_{s=1}^r I_s = \mathbb{N}$ with the N-th group of indices corresponding to the eigenfunctions $\{x_m f_N\}_{1\le m\le N}$, we obtain a partition $\bigcup_{s=1}^r I_{s,N} = I^N$ of the set $I^N = \{1, 2, \ldots, N\}$, where index m runs. Reasoning by induction, assume we have an infinite subsequence $\{N_i\}$ of $\mathbb{N}$ such that for a given N all partitions $\bigcup_{s=1}^r (I_{s,N_i} \bigcap I^N) = I^N$, $i \ge 1$, are the same. Since there is only a finite

number of partitions of I^{N+1}, we can choose an infinite subsequence of $\{N_i\}$, say $\{N_l'\}$, such that all partitions $\bigcup_{s=1}^r (I_{s,N_l'} \bigcap I^{N+1}) = I^{N+1}$, $l \geq 1$, are the same. Applying a diagonal process to this table of sequences, we obtain a growing sequence of integers $\{M_i\}_{i\geq 1}$ such that all partitions $\bigcup_{s=1}^r (I_{s,M_i} \bigcap I^N) = I^N$, $i \geq 1$, are the same, for all $N = 1, 2, \ldots$. This means that we have a partition $\bigcup_{s=1}^r \sigma_s = \mathbb{N}$, $I_{s,M_i} \bigcap I^N = \sigma_s \bigcap I^N$. Next, we observe that, for every s, $1 \leq s \leq r$,

$$\begin{aligned} 0 < \delta &:= \inf\{\|Ax\| \colon x \in H_{I_s}, \|x\| = 1\} \\ &\leq \inf\{\|T_N f\| \colon f \in H_{I_{s,M_i} \cap I^N}, \|f\| = 1\} \end{aligned}$$

for every $N \geq 1$. Taking $N \to \infty$, we get

$$0 < \delta \leq \inf\{\|Tf\| \colon f \in H_{\sigma_s}, \|f\| = 1\}$$

for every s, $1 \leq s \leq r$.

But, as we saw above, this is impossible. □

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